

# A NON-EXISTENCE RESULT FOR A GENERALIZATION OF THE EQUATIONS OF THE CONFORMAL METHOD IN GENERAL RELATIVITY

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**ABSTRACT.** The constraint equations of general relativity can in many cases be solved by the conformal method. We show that a slight modification of the equations of the conformal method admits no solution for a broad range of parameters. This suggests that the question of existence or non-existence of solutions to the original equations is more subtle than could perhaps be expected.

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## 1. INTRODUCTION

Initial data for the Cauchy problem in general relativity consists of a Riemannian manifold  $(M, \tilde{g})$  and a symmetric 2-tensor  $\tilde{K}$  on  $M$  satisfying the following equations,

$$R^{\tilde{g}} - |\tilde{K}|_{\tilde{g}}^2 + (\operatorname{tr}^{\tilde{g}} \tilde{K})^2 = 0, \quad (1a)$$

$$\operatorname{div}^{\tilde{g}} \tilde{K} - d \operatorname{tr}^{\tilde{g}} \tilde{K} = 0. \quad (1b)$$

Here  $\tilde{g}$  represents the metric induced by the space-time metric on the Cauchy surface  $M$ ,  $\tilde{K}$  is its second fundamental form, and  $R^{\tilde{g}}$  is the scalar curvature of the metric  $\tilde{g}$ . The constraint equations (1) follow from the vacuum Einstein equation for the space-time metric.

Constructing and classifying solutions of this system is an important issue. Background and many results are summarized in the excellent review article [2]. One of the most important methods to construct solutions to this system is the conformal method. Its main idea is to choose part of the initial data as given and then solve for the rest of the data.

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For the given data of an  $n$ -dimensional Riemannian manifold  $(M, g)$ ,  $n \geq 3$ , a function  $\tau$  on  $M$  and a symmetric traceless divergence-free 2-tensor  $\sigma$  on  $M$ , one seek for a positive function  $\varphi$  and a 1-form  $W$  on  $M$  such that

$$\tilde{g} = \varphi^{N-2}g, \quad \tilde{K} = \frac{\tau}{n}\tilde{g} + \varphi^{-2}(\sigma + LW),$$

solve the constraint equations (1). Here we have set  $N := \frac{2n}{n-2}$ . Further,  $L$  denotes the conformal Killing operator acting on 1-forms. In coordinates

$$(LW)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla^k W_k g_{ij},$$

where  $\nabla$  is the Levi-Civita connection associated to the metric  $g$ . Note that  $\tau$  corresponds to the mean curvature of  $M$  embedded in the space-time solving Einstein's equations.

The constraint equations are satisfied if  $\varphi$  and  $W$  solve the system

$$\frac{4(n-1)}{n-2} \Delta \varphi + R\varphi = -\frac{n-1}{n} \tau^2 \varphi^{N-1} + |\sigma + LW|^2 \varphi^{-N-1}, \quad (2a)$$

$$-\frac{1}{2} L^* LW = \frac{n-1}{n} \varphi^N d\tau, \quad (2b)$$

where  $\Delta$  is the non-negative Laplacian and  $L^*$  is the formal  $L^2$ -adjoint of  $L$ . As with all other metric-dependent objects used from here on they are defined using the metric  $g$ . The first equation is known as the Lichnerowicz equation while the second is called the vector equation.

Of special importance is the case where  $\tau$  is constant, so that the corresponding Cauchy surface has constant mean curvature in the surrounding space-time. Making this assumption renders the system much simpler to solve. Indeed, Equation (2b) becomes  $L^* LW = 0$ . Under the generic condition that  $(M, g)$  has no conformal Killing vector field this implies that  $W = 0$ , and it remains only to solve the Lichnerowicz equation (2a).

Hence much work has been devoted to the study of the cases when  $\tau$  is constant or when  $d\tau$  small. We refer the reader to [2] and references therein for more details. However, as proven in [1], such a Cauchy surface of constant mean curvature does not exist for all space-times.

Recent new results were obtained for the case of arbitrary  $\tau$  by M. Holst, G. Nagy, G. Tsogtgerel in [5, 6], by D. Maxwell in [8], by the authors in [3], and by the second author with A. Sakovich in [4]. In [9], D. Maxwell studied a simplified model problem to get insight into the solvability of the system (2).

Non-existence results for the system (2) are very rare, see for example [7, Theorem 2] and [3, Theorem 1.7] where  $\sigma \equiv 0$  is assumed. The purpose of this short note is to provide an example of a system very similar to the original system (2) which does not admit any solutions as soon as a certain parameter is larger than a fairly explicit constant. We will prove the following theorem.

**Theorem 1.1.** *Let  $(M, g)$  be a Riemannian manifold of dimension  $3 \leq n \leq 5$  with scalar curvature  $R \leq 0$ ,  $R \not\equiv 0$ . Let  $\tau$  be a function and  $\sigma$  a symmetric traceless divergence-free 2-tensor on  $M$ . Assume that  $\xi$  is a  $C^1$  1-form which does not vanish anywhere on  $M$ . Then there is a constant  $a_0$  such that there does not*

exist any solution to the system

$$\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1} + |\sigma + LW|^2\varphi^{-N-1}, \quad (3a)$$

$$-\frac{1}{2}L^*LW = a\varphi^N\xi, \quad (3b)$$

when  $a > a_0$ .

The constant  $a_0$  depends on a Sobolev constant, a constant appearing in a Schauder estimate,  $\max|\xi|$ ,  $\min|\xi|$ ,  $\|L\xi\|_{L^2}$ ,  $\|L\xi\|_{L^\infty}$ ,  $\|\tau\|_{L^{2N}}$ ,  $\min\tau$ ,  $\|R\|_{L^n}$ ,  $\|\sigma\|_{L^N}$  and on  $\|\varphi_-\|_{L^N}$ , where  $\varphi_- > 0$  is the solution of the prescribed scalar curvature equation

$$\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi = -\frac{n-1}{n}\tau^2\varphi^{N-1}.$$

By the assumption on scalar curvature the metric  $g$  has negative Yamabe constant which guarantees the existence of the function  $\varphi_-$ .

Note that the only difference between the modified system (3) and the original system (2) is that  $(n-1)d\tau$  in (2b) is replaced by  $a\xi$ .

This theorem shows that if a general existence result (that is without additional assumption on  $\tau$  and  $\sigma$ ) for the system (2) exists, then its proof must use the fact that  $d\tau$  (appearing in (2b)) is the differential of  $\tau$  and not an arbitrary 1-form. Even more, this seems to indicate that such a general statement could be false.

The theorem stated here is certainly not in the most general form possible, our goal is simply to find an example of a non-existence result for the generalized system (3).

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## 2. PROOF OF THEOREM 1.1

Assume that  $(\varphi, W)$  solves the system (3). We define

$$\gamma := \int_M |\sigma + LW|^2 d\mu$$

and rescale Equations (3a)-(3b) by setting

$$\tilde{\varphi} := \frac{1}{\gamma^{\frac{1}{2N}}}\varphi, \quad \tilde{W} := \frac{1}{\gamma^{\frac{1}{2}}}W, \quad \tilde{\sigma} := \frac{1}{\gamma^{\frac{1}{2}}}\sigma.$$

The equations then become

$$\frac{1}{\gamma^{\frac{1}{n}}}\left(\frac{4(n-1)}{n-2}\Delta\tilde{\varphi} + R\tilde{\varphi}\right) = -\frac{n-1}{n}\tau^2\tilde{\varphi}^{N-1} + \left|\tilde{\sigma} + L\tilde{W}\right|^2\tilde{\varphi}^{-N-1}, \quad (4a)$$

$$-\frac{1}{2}L^*L\tilde{W} = a\tilde{\varphi}^N\xi. \quad (4b)$$

The proof of Theorem 1.1 is decomposed in a sequence of claims.

**Claim 2.1.** *There exists a constant  $c_1 > 0$  such that*

$$\gamma \geq c_1 a^2.$$

We remark that  $\varphi \geq \varphi_-$ , see for example [3]. Taking the scalar product of (3b) with  $\xi$  and integrating we get the central equality of the following estimate,

$$\begin{aligned}
\frac{1}{2} \left( \int_M |L\xi|^2 d\mu \right)^{\frac{1}{2}} \left( \int_M |LW + \sigma|^2 d\mu \right)^{\frac{1}{2}} &\geq -\frac{1}{2} \int_M \langle L\xi, LW + \sigma \rangle d\mu \\
&= -\frac{1}{2} \int_M \langle L\xi, LW \rangle d\mu \\
&= a \int_M \varphi^N |\xi|^2 d\mu \\
&\geq a \left( \inf_M |\xi|^2 \right) \int_M \varphi^N d\mu \\
&\geq a \left( \inf_M |\xi|^2 \right) \int_M \varphi_-^N d\mu,
\end{aligned}$$

from which we conclude

$$\gamma^{\frac{1}{2}} \geq 2 \frac{(\inf_M |\xi|^2) \int_M \varphi_-^N d\mu}{\left( \int_M |L\xi|^2 d\mu \right)^{\frac{1}{2}}} a.$$

The claim thus holds with

$$c_1 := 4 \frac{(\inf_M |\xi|^2)^2 \left( \int_M \varphi^N d\mu \right)^2}{\int_M |L\xi|^2 d\mu}.$$

**Claim 2.2.** *There exists a constant  $c_2$  such that*

$$\int_M \tilde{\varphi}^N d\mu \leq \frac{c_2}{a^{N+1}}.$$

We take the scalar product of (4b) with  $\xi$  and integrate over  $M$  to find

$$\begin{aligned}
\frac{1}{2} \|L\xi\|_{L^\infty} \left( \int_M |L\tilde{W} + \tilde{\sigma}|^2 \tilde{\varphi}^{-N} d\mu \right)^{\frac{1}{2}} \left( \int_M \tilde{\varphi}^N d\mu \right)^{\frac{1}{2}} \\
&\geq \frac{1}{2} \|L\xi\|_{L^\infty} \int_M |L\tilde{W} + \tilde{\sigma}| d\mu \\
&\geq -\frac{1}{2} \int_M \langle L\xi, L\tilde{W} + \tilde{\sigma} \rangle d\mu \\
&= -\frac{1}{2} \int_M \langle L\xi, L\tilde{W} \rangle d\mu \\
&= a \int_M \tilde{\varphi}^N |\xi|^2 d\mu \\
&\geq a \left( \inf_M |\xi|^2 \right) \int_M \tilde{\varphi}^N d\mu
\end{aligned}$$

or

$$\frac{1}{2} \|L\xi\|_{L^\infty} \left( \int_M |L\tilde{W} + \tilde{\sigma}|^2 \tilde{\varphi}^{-N} d\mu \right)^{\frac{1}{2}} \geq a \left( \inf_M |\xi|^2 \right) \left( \int_M \tilde{\varphi}^N d\mu \right)^{\frac{1}{2}}.$$

From the Hölder inequality we get

$$\begin{aligned} & \int_M \left| L\widetilde{W} + \widetilde{\sigma} \right|^2 \widetilde{\varphi}^{-N} d\mu \\ & \leq \left( \int_M \left| L\widetilde{W} + \widetilde{\sigma} \right|^2 \widetilde{\varphi}^{-N-1} d\mu \right)^{\frac{N}{N+1}} \left( \int_M \left| L\widetilde{W} + \widetilde{\sigma} \right|^2 d\mu \right)^{\frac{1}{N+1}} \\ & \leq \left( \int_M \left| L\widetilde{W} + \widetilde{\sigma} \right|^2 \widetilde{\varphi}^{-N-1} d\mu \right)^{\frac{N}{N+1}}, \end{aligned}$$

and together with the previous estimate we have found that

$$\left( \int_M \widetilde{\varphi}^N d\mu \right)^{\frac{1}{2}} \leq \frac{1}{2a} \frac{\|L\xi\|_{L^\infty}}{(\inf_M |\xi|^2)} \left( \int_M \left| L\widetilde{W} + \widetilde{\sigma} \right|^2 \widetilde{\varphi}^{-N-1} d\mu \right)^{\frac{N}{2(N+1)}}. \quad (5)$$

To estimate the right-hand side we integrate Equation (4a),

$$\frac{1}{\gamma^{\frac{1}{n}}} \int_M R\widetilde{\varphi} d\mu + \frac{n-1}{n} \int_M \tau^2 \widetilde{\varphi}^{N-1} d\mu = \int_M \left| \widetilde{\sigma} + L\widetilde{W} \right|^2 \widetilde{\varphi}^{-N-1} d\mu.$$

Since  $R \leq 0$  we see that

$$\begin{aligned} \int_M \left| \widetilde{\sigma} + L\widetilde{W} \right|^2 \widetilde{\varphi}^{-N-1} d\mu & \leq \frac{n-1}{n} \int_M \tau^2 \widetilde{\varphi}^{N-1} d\mu \\ & \leq \frac{n-1}{n} \left( \int_M \tau^{2N} d\mu \right)^{\frac{1}{N}} \left( \int_M \widetilde{\varphi}^N d\mu \right)^{\frac{N-1}{N}}. \end{aligned} \quad (6)$$

Inserting this estimate into (5) we get

$$\begin{aligned} \left( \int_M \widetilde{\varphi}^N d\mu \right)^{\frac{1}{2}} & \leq \frac{1}{2a} \left( \frac{n-1}{n} \right)^{\frac{N}{2(N+1)}} \frac{\|L\xi\|_{L^\infty}}{(\inf_M |\xi|^2)} \left( \int_M \tau^{2N} d\mu \right)^{\frac{1}{2(N+1)}} \\ & \quad \cdot \left( \int_M \widetilde{\varphi}^N d\mu \right)^{\frac{N-1}{2(N+1)}}, \end{aligned}$$

or

$$\int_M \widetilde{\varphi}^N d\mu \leq \frac{1}{(2a)^{N+1}} \left( \frac{n-1}{n} \right)^{\frac{N}{2}} \left( \frac{\|L\xi\|_{L^\infty}}{(\inf_M |\xi|^2)} \right)^{N+1} \left( \int_M \tau^{2N} d\mu \right)^{\frac{1}{2}},$$

which proves the claim with

$$c_2 := \frac{1}{2^{N+1}} \left( \frac{n-1}{n} \right)^{\frac{N}{2}} \left( \frac{\|L\xi\|_{L^\infty}}{(\inf_M |\xi|^2)} \right)^{N+1} \left( \int_M \tau^{2N} d\mu \right)^{\frac{1}{2}}.$$

As a corollary, plugging Claim 2.2 into (6), we get

**Claim 2.3.** *There exists a constant  $c_3$  such that*

$$\int_M \left| \widetilde{\sigma} + L\widetilde{W} \right|^2 \widetilde{\varphi}^{-N-1} d\mu \leq \frac{c_3}{a^{\frac{(N+1)(N-1)}{N}}}.$$

**Claim 2.4.** *For a large enough there exists a constant  $c_4$  such that*

$$\int_M \widetilde{\varphi}^{2N} d\mu \leq c_4.$$

We multiply (4a) by  $\tilde{\varphi}^{N+1}$  and integrate over  $M$ ,

$$\begin{aligned}
& \frac{1}{\gamma^{\frac{1}{n}}} \int_M \left( \frac{4(n-1)}{n-2} \frac{N+1}{(\frac{N}{2}+1)^2} \left| d\tilde{\varphi}^{\frac{N}{2}+1} \right|^2 + R\tilde{\varphi}^{N+2} \right) d\mu \\
&= \frac{1}{\gamma^{\frac{1}{n}}} \int_M \left( \frac{4(n-1)}{n-2} \tilde{\varphi}^{N+1} \Delta \tilde{\varphi} + R\tilde{\varphi}^{N+2} \right) d\mu \\
&= -\frac{n-1}{n} \int_M \tau^2 \tilde{\varphi}^{2N} d\mu + \int_M \left| \tilde{\sigma} + L\tilde{W} \right|^2 d\mu \\
&= -\frac{n-1}{n} \int_M \tau^2 \tilde{\varphi}^{2N} d\mu + 1,
\end{aligned}$$

from which we conclude that

$$\frac{1}{\gamma^{\frac{1}{n}}} \int_M R\tilde{\varphi}^{N+2} d\mu + \frac{n-1}{n} (\inf_M \tau)^2 \int_M \tilde{\varphi}^{2N} d\mu \leq 1.$$

Using Young's inequality we infer

$$\begin{aligned}
\left| \int_M R\tilde{\varphi}^{N+2} d\mu \right| &\leq \frac{N+2}{2N} \int_M \tilde{\varphi}^{2N} d\mu + \frac{N-2}{2N} \int_M |R|^{\frac{2N}{N-2}} d\mu \\
&= \frac{n-1}{n} \int_M \tilde{\varphi}^{2N} d\mu + \frac{1}{n} \int_M |R|^{\frac{2N}{N-2}} d\mu,
\end{aligned}$$

so

$$\frac{n-1}{n} \left( (\inf_M \tau)^2 - \frac{1}{\gamma^{\frac{1}{n}}} \right) \int_M \tilde{\varphi}^{2N} d\mu \leq 1 + \frac{1}{n\gamma^{\frac{1}{n}}} \int_M |R|^{\frac{2N}{N-2}} d\mu.$$

Assuming that  $\gamma^{-\frac{1}{n}} \leq \frac{1}{2} (\inf_M \tau)^2$  which by Claim 2.1 holds for large enough  $a$  the claim follows with

$$c_4 := \frac{2n}{n-1} \frac{1}{(\inf_M \tau)^2} \left( 1 + \frac{2}{n(\inf_M \tau)^2} \int_M |R|^{\frac{2N}{N-2}} d\mu \right).$$

**Claim 2.5.** *If  $a$  is large enough, there exists a constant  $c_5$  such that*

$$\int_M \left| \tilde{\sigma} + L\tilde{W} \right|^N d\mu \leq c_5 a^N.$$

From standard elliptic regularity and Sobolev injection there exists a constant  $C$  such that

$$\begin{aligned}
\|L\tilde{W}\|_{L^N}^2 &\leq C \int_M \left| -\frac{1}{2} L^* L\tilde{W} \right|^2 d\mu \\
&\leq C a^2 \int_M \tilde{\varphi}^{2N} |\xi|^2 d\mu \\
&\leq C (\sup_M |\xi|)^2 c_4 a^2,
\end{aligned}$$

where we also used Claim 2.4. Hence,

$$\begin{aligned}
\int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^N d\mu &= \left\| \tilde{\sigma} + L\widetilde{W} \right\|_{L^N}^N \\
&\leq \left( \|\tilde{\sigma}\| + \left\| L\widetilde{W} \right\|_{L^N} \right)^N \\
&\leq 2^N \left( \|\tilde{\sigma}\|_{L^N}^N + \left\| L\widetilde{W} \right\|_{L^N}^N \right) \\
&\leq 2^N \left( \|\tilde{\sigma}\|_{L^N}^N + C^{\frac{N}{2}} (\sup_M |\xi|)^N c_4^{\frac{N}{2}} a^N \right).
\end{aligned}$$

This proves the claim with

$$c_5 := 2^{N+1} C^{\frac{N}{2}} (\sup_M |\xi|)^N c_4^{\frac{N}{2}}.$$

**Claim 2.6.** *There exists a constant  $c_6$  such that*

$$\int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{N-2} d\mu \leq \frac{c_6}{a^{(N+1)\frac{N-2}{N}-2}}.$$

From Hölder's inequality with Claims 2.2 and 2.5 we get

$$\begin{aligned}
\int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{N-2} d\mu &\leq \left( \int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^N d\mu \right)^{\frac{2}{N}} \left( \int_M \tilde{\varphi}^N d\mu \right)^{\frac{N-2}{N}} \\
&\leq c_5^{\frac{2}{N}} a^2 \frac{c_2^{\frac{N-2}{N}}}{a^{(N+1)\frac{N-2}{N}}},
\end{aligned}$$

which proves the claim.

**Claim 2.7.** *There exists a constant  $c_7$  such that*

$$\int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{-N+2} d\mu \leq \frac{c_7}{a^{\frac{(N-2)(N-1)}{N}}}$$

Using Hölder's inequality and Claim 2.3 we find

$$\begin{aligned}
&\int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{-N+2} d\mu \\
&\leq \left( \int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 d\mu \right)^{\frac{3}{N+1}} \left( \int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{-N-1} d\mu \right)^{\frac{N-2}{N+1}} \\
&\leq \frac{c_3^{\frac{N-2}{N+1}}}{a^{\frac{(N-2)(N-1)}{N}}},
\end{aligned}$$

which is the statement of the claim.

We are now ready to prove Theorem 1.1. For this we use the Hölder inequality with Claims 2.6 and 2.7 to get

$$\begin{aligned}
1 &= \left( \int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 d\mu \right)^2 \\
&\leq \int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{N-2} d\mu \int_M \left| \tilde{\sigma} + L\widetilde{W} \right|^2 \tilde{\varphi}^{-N+2} d\mu \\
&\leq c_6 c_7 a^{-(N+1)\frac{N-2}{N}+2-\frac{(N-2)(N-1)}{N}} \\
&\leq c_6 c_7 a^{6-2N}.
\end{aligned} \tag{7}$$

The exponent of  $a$  is negative if and only if  $3 \leq n \leq 5$ . Hence if  $a$  is large enough the inequality (7) gives a contradiction.

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